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On dense-lineability of sets of functions on $\mathbb{R}$

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ABSTRACT

A subset M of a topological vector space X is said to be dense-lineable in X if there exists an infinite dimensional linear manifold in $M \cup \{0\}$ and dense in X. We give sufficient conditions for a lineable set to be dense-lineable, and we apply them to prove the dense-lineability of several subsets of $\mathcal{C}[a, b]$. We also develop some techniques to show that the set of differentiable nowhere monotone functions is dense-lineable in $\mathcal{C}[a, b]$. Other results related to density and dense-lineability of sets in Banach spaces are also presented.

1. Preliminaries and background

A set M of functions enjoying some special property is said to be *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space, and *spaceable* if $M \cup \{0\}$ contains a closed infinite dimensional linear manifold. Also, we let $\lambda(M)$ be the maximum dimension (if it exists) of such a manifold. The terminology of lineable and spaceable was introduced by Gurariy in [1,2] (see [3] as well). Gurariy also coined the following concept (which was also used in [4] under the name of *algebraic genericity*):

Definition 1.1 (*[1,2]*). Let M be a subset of a topological vector space X. M is said to be dense-lineable in X if there exists an infinite dimensional linear manifold $Y \subset M \cup \{0\}$ dense in X.

For example, this concept has been studied in [5], where for any set $E \subset \mathbb{T}$ of Lebesgue measure zero, the authors construct a dense infinitely generated algebra (and, therefore, a dense infinite dimensional linear manifold) of continuous functions whose Fourier series expansion is divergent at any point $t \in E$.

In Section 2, we introduce the concept of *strong sets* in a Banach space and use it to provide new proofs of some results concerning dense-lineability. In Section 3, we use the idea of the "join" of two functions defined on subintervals of a given interval. Using this, we prove the density of the set $DNM[a, b]$ of differentiable nowhere monotone functions in the Banach space $\mathcal{C}[a, b]$. (Note that such counter-intuitive functions exist, as can be seen, for example, in [6].) Finally, in Section 4, we prove the dense-lineability of $DNM[a, b]$ in $\mathcal{C}[a, b]$.

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2. Strong sets in vector spaces and dense-lineability in Banach spaces

2.1. General results on dense-lineability in Banach spaces

In this section we will study necessary conditions on a lineable set to be dense-lineable. In order to do that we will need to introduce the notion of *strong set*.

Definition 2.1. Let A, B be subsets of a vector space X. We say that A is stronger than B if $A + B \subseteq A$.

Note that in general, $0 \notin A$. The following result will be give us a technique to prove the dense-lineability of certain lineable sets.

Theorem 2.2. *Let X be a separable Banach space, and consider two subsets A, B of X such that A is lineable and B dense-lineable. If A is stronger than B, then A is dense-lineable.*

Proof. There exist infinite dimensional vector subspaces Y and Z of X such that $Y \subset A \cup \{0\}$, $Z \subset B \cup \{0\}$, and Z is dense. Let us take an infinite linearly independent family $\{y_n : n \in \mathbb{N}\}$ of Y with $\|y_n\| = 1$ for every $n \in \mathbb{N}$. Let $\{z_n : n \in \mathbb{N}\}$ be a dense sequence contained in Z. Consider now

$$W = \operatorname{span}\left\{\frac{1}{n}y_n + z_n : z \in \mathbb{N}\right\}.$$

W is our "candidate" as a dense subspace contained in A. First, notice that $W \subset A \cup \{0\}$. Indeed, if $\lambda_1, \ldots, \lambda_p \in \mathbb{R} \setminus \{0\}$ and $n_1, \ldots, n_p$ are different natural numbers, then

$$\lambda_1 \cdot \left(\frac{1}{n_1}y_{n_1} + z_{n_1}\right) + \cdots + \lambda_p \cdot \left(\frac{1}{n_p}y_{n_p} + z_{n_p}\right) = \sum_{j=1}^{p} \frac{\lambda_j}{n_j} y_{n_j} + \sum_{j=1}^{p} \lambda_j z_{n_j} \in (Y \setminus \{0\}) + Z \subset A.$$

Also, W is dense in X. This follows from the construction, since $\{z_n : n \in \mathbb{N}\}$ was chosen dense, so $\{\frac{1}{n}y_n + z_n : z \in \mathbb{N}\}$ is dense as well. $\square$

On the other hand, the following example shows us that dense-lineability does not occur as often as one might expect.

Example 2.3. Let X be an infinite dimensional Banach space. There exists a subset $M \subset X$ such that M is spaceable and dense, although it is not dense-lineable.

Proof. Let us consider any proper infinite dimensional closed subspace Y of X. Let Z be an algebraic complement for Y in X, and let $\mathcal{B}$ be a Hamel basis for Z. Next, take

$$M = Y + \operatorname{span}_{\mathbb{Q}} \mathcal{B},$$

where $\operatorname{span}_{\mathbb{Q}} \mathcal{B}$ denotes the $\mathbb{Q}$-linear span of $\mathcal{B}$.

We will now show that M enjoys the required properties. Since $Y \subset M$, M is spaceable. M is also dense. Indeed, $\operatorname{span}_{\mathbb{Q}} \mathcal{B}$ is dense in Z, so $M = Y + \operatorname{span}_{\mathbb{Q}} \mathcal{B}$ is dense in $Y + Z = X$.

M is not dense-lineable. Indeed, suppose that W is a dense vector subspace of X contained in $M \cup \{0\} = M$. Take $w \in W$ and write

$$w = y + q_1 b_1 + \cdots + q_n b_n$$

with $y \in Y$, $q_1, \ldots, q_n \in \mathbb{Q}$, and $b_1, \ldots, b_n \in \mathcal{B}$. Our aim is to show that $q_j = 0$ for every $j \in \{1, 2, \ldots, n\}$. Suppose not, assuming that $q_1 \neq 0$. Then $\frac{\pi}{q_1} w \in W \subset M$; that is,

$$\frac{\pi}{q_1} w = y' + q_1' b_1' + \cdots + q_{n'}' b_{n'}',$$

with $y' \in Y$, $q_1', \ldots, q_{n'}' \in \mathbb{Q}$, and $b_1', \ldots, b_{n'}' \in \mathcal{B}$. We therefore have that

$$y' = \frac{\pi}{q_1} y,$$

and

$$\pi b_1 + \frac{q_2 \pi}{q_1} b_2 + \cdots + \frac{q_n \pi}{q_1} b_n = q_1' b_1' + \cdots + q_{n'}' b_{n'}',$$

obtaining the contradiction that $\pi \in \mathbb{Q}$. Thus, all q_j's are 0. Finally, $W \subset Y$, which (again) is a contradiction because Y is closed and proper. $\square$

Notice that, in the previous example, M can be chosen so that $\lambda(M) = \dim(X)$. Indeed, it suffices to take Y as the kernel of any nonzero continuous functional. The next result is a counter-part to the previous one, and allows us to conclude that there is no relation between spaceability and dense-lineability.

Example 2.4. Let X be an infinite dimensional Banach space. There exists a subset $M \subset X$ which is lineable and dense, but which is not spaceable. If X is separable, then M can also be chosen to be dense-lineable.

Proof. Let us consider a Hamel basis $\mathcal{B}$ for X and a countably infinite subset $\{b_n : n \in \mathbb{N}\}$ of $\mathcal{B}$. We write $X = Y \oplus Z$ where $Y = \text{span}\,\{b_n : n \in \mathbb{N}\}$ and $Z = \text{span}\,(\mathcal{B} \setminus \{b_n : n \in \mathbb{N}\})$. Our candidate will be

$$M = Y + \text{span}_{\mathbb{Q}}\,(\mathcal{B} \setminus \{b_n : n \in \mathbb{N}\})\,.$$

We claim that M satisfies the required properties. M is lineable, since $Y \subset M$. M is also dense. To see this, note that $\text{span}_{\mathbb{Q}}\,(\mathcal{B} \setminus \{b_n : n \in \mathbb{N}\})$ is dense in Z, so $M = Y + \text{span}_{\mathbb{Q}}\,(\mathcal{B} \setminus \{b_n : n \in \mathbb{N}\})$ is dense in $Y + Z = X$.

Finally, M is not spaceable. Indeed, suppose that W is an infinite dimensional closed vector subspace of X contained in $M \cup \{0\} = M$. Then, by proceeding as we did in the previous example, we deduce that $W \subset Y$, but this is impossible because the cardinality of any Hamel basis of W is uncountable.

To finish the proof, notice that if X is separable, then we can choose Y to be dense in X, and therefore M is dense-lineable. □

Remark 2.5. We would like to point out that Theorem 2.2 can be extended to metrizable topological vector spaces that are not normable, as we will notice after a careful reading of Section 2.2.3. The same can also be said for Examples 2.3 and 2.4.

2.2. Examples of strong properties and dense-lineability

We list below some examples of how Theorem 2.2 can be used to obtain dense-lineability in many cases in an almost trivial way.

2.2.1. Continuous nowhere differentiable functions

Here, we apply Theorem 2.2 to prove the following result (an independent proof of a more general result can be found in [7] and also in [8]):

Theorem 2.6. *The set $ND[a, b]$ of continuous nowhere differentiable functions in $[a, b]$ is dense-lineable in $\mathcal{C}[a, b]$.*

Proof. Without loss, we consider the interval $[a, b]$ to be the interval $[0, 1]$. Using the fact that $ND[0, 1]$ is lineable (see [9]), let us consider the dense-linear manifold of all polynomials in $\mathcal{C}[0, 1]$, and call this set $\mathcal{P}$. Clearly, $ND[0, 1]$ is stronger than $\mathcal{P}$, therefore (by Theorem 2.2) $ND[0, 1]$ is dense-lineable. □

2.2.2. $\mathcal{C}^\infty$ non-analytic functions

Theorem 2.7. *For any interval $[a, b]$, the set of non-analytic $\mathcal{C}^\infty$ functions, as well as the set of functions that are in $\mathcal{C}^m \setminus \mathcal{C}^n$ (with $m < n$), are dense-lineable.*

Proof. It is enough to take the interval $[-1, 1]$. Here we consider the infinite dimensional subspace S of $\mathcal{C}^\infty[-1, 1]$ given by

$$S = \text{span}\{e^{-\frac{\alpha}{x^2}} : \alpha > 0\}$$

(cf [10]). It is clear that every function in $S \setminus \{0\}$ is non-analytic, since all derivatives at 0 are equal to 0. However the set of $\mathcal{C}^\infty$ non-analytic functions is clearly stronger than the set of polynomials, which is dense. An appeal to Theorem 2.2 yields the first result.

For the second one we take an interval $[a, b]$, $m < n$ and a positive function $f \in \mathcal{C}^m[a, b] \setminus \mathcal{C}^n[a, b]$. We can consider the infinite dimensional linear manifold

$$S = \text{span}\{e^{\alpha x} f(x) : \alpha > 0\} \subset \mathcal{C}^m[a, b] \setminus \mathcal{C}^n[a, b].$$

The density of the polynomials, the trivial fact that $\mathcal{C}^m[a, b] \setminus \mathcal{C}^n[a, b]$ is stronger than the polynomials, and Theorem 2.2 finish the proof. □

Remark 2.8. We would like to point out that stronger results than the one appearing in Theorem 2.7 can be found in [8,11].

2.2.3. Universal functions for MacLane's operator

In 1952, MacLane [12] constructed a universal entire function for the differentiation operator

$$\begin{aligned} D \quad : \quad & \mathcal{H}(\mathbb{C}) \longrightarrow \mathcal{H}(\mathbb{C}) \\ & f(z) \mapsto f'(z). \end{aligned}$$

This means that there exists an entire function f such that the set $\{f^{(n)} : n = 1, 2, \ldots\}$ is dense in $\mathcal{H}(\mathbb{C})$, endowed with the compact-open topology.

It is known that there exists a dense manifold of universal functions for the operator D [13]. (In fact, any universal operator on any topological vector space admits a dense-linear manifold consisting, except for zero, of universal vectors; see [14].)

Remark 2.9. Let us here, and for the special case of the operator D, include a proof of the fact that the set of universal functions for D is lineable. Take a universal entire function f. Then $f', f'', f''', \ldots$ are also universal. But the elements of the orbit of a T-universal vector under a universal operator T are linearly independent (since a finite dimensional topological vector space cannot support a universal operator). Hence the linear span S of $\{f', f'', f''', \ldots\}$ is infinite dimensional. Moreover, all nonzero differential polynomials in f are D-universal, because $P(D)f$ is universal for every nonzero polynomial P; indeed,

$$\{D^n P(D)f : n = 1, 2, \ldots\} = P(D)(\{D^n f : n = 1, 2, \ldots\})$$

and $P(D)$ is surjective (it can be decomposed into finitely many differentiable operators of order 1).

Now, we can show that the set of universal functions for D is dense-lineable by means of an slight modification of Theorem 2.2, which also holds in Fréchet spaces. Indeed, let us observe that, in the proof of Theorem 2.2, the infinite linearly independent family $\{y_n : n \in \mathbb{N}\}$ chosen in Y needs to be bounded, and this can be accomplished in Fréchet spaces as follows. There exists a countable local basis $\{U_n : n \in \mathbb{N}\}$ of (convex) balanced absorbing neighborhoods of 0 in Y. Therefore, if $\left\{y'_n : n \in \mathbb{N}\right\}$ is any infinite linearly independent family in Y, for each $n \in \mathbb{N}$ there exists $\lambda_n > 0$ with $\lambda_n y'_n \in U_n$. Now, $\left\{y_n := \lambda_n y'_n : n \in \mathbb{N}\right\}$ is an infinite linearly independent family of Y converging to 0 and hence is bounded.

We can now use the lineability of the set of universal functions for D, call it $HC(D)$, in order to show the dense-lineability of $HC(D)$.

Theorem 2.10. *The set $HC(D)$ is dense-lineable.*

Proof. It suffices to consider the set of all polynomials, $\mathcal{P}$, which is dense in $\mathcal{H}(\mathbb{C})$. Clearly, $HC(D)$ is stronger than $\mathcal{P}$. The result then follows from the above remarks on Theorem 2.2. □

2.2.4. The set $L^p(I) \setminus L^q(I)$

In [15], it was shown that if $1 \leq p < q$, then the set $L^p[0, 1] \setminus L^q[0, 1]$ is lineable. Clearly, if $\mathcal{P}$ denotes the set of polynomials, which is dense in $L^p[0, 1]$, then we have that $L^p[0, 1] \setminus L^q[0, 1]$ is stronger than $\mathcal{P}$. We obtain the following.

Theorem 2.11. *The set $L^p[0, 1] \setminus L^q[0, 1]$ is dense-lineable in $L^p[0, 1]$.*

Similarly, it can also be proved that if I is any unbounded interval, then $L^p(I)\setminus L^q(I)$ is also dense-lineable in $L^p(I)$ whenever $p > q \geq 1$.

2.2.5. Functions whose derivative is unbounded on a closed interval

The authors showed in [10] that the set UD of functions on $\mathbb{R}$ whose derivative is unbounded on a closed interval is lineable. As before, UD is stronger than $\mathcal{P}$, and so we have the next result.

Theorem 2.12. *The set of functions whose derivative is unbounded on an arbitrary closed interval I is dense-lineable in $\mathcal{C}(I)$.*

3. Joins, chains, and density of $DNM[a, b]$ in $\mathcal{C}[a, b]$

In this section, we focus on showing the denseness and, in fact, the dense-lineability of the set $DNM[a, b]$ of differentiable, nowhere monotone functions on a closed interval $[a, b]$ in $\mathcal{C}[a, b]$. In order to do this, we begin with a description of the concepts that will be used in this section.

3.1. Definitions: Joins and chains

We will be interested in a pair of the form $J = ([a, b], f)$, where $f \in \mathcal{C}[a, b]$. Assuming that $f(a) \neq f(b)$, the *rectangularity* of $J = ([a, b], f)$ is the real number $r(J) \in [1, \infty)$ defined by

$$r(J) := \frac{\mathrm{Osc}_{[a,b]} f}{|f(b) - f(a)|},$$

where $\mathrm{Osc}_{[a,b]} f$ is the oscillation of f in $[a, b]$, that is, the value

$$\sup\{|f(x) - f(y)| : x, y \in [a, b]\}.$$

If $f(a) = f(b)$ we define $r(J) = 1$.

If $r(J) \leq k$, we say that $([a, b], f)$ is *k-rectangular*. Two such pairs $J_1 = ([a, b], f)$ and $J_2 = ([b, c], g)$ will be called *connected* if $f(b) = g(b)$. In such a case, we can form their *join* $J_1 \odot J_2$, defined on $[a, c]$ in the obvious way. Clearly, we can continue this process, forming joins $J_1 \odot J_2 \odot J_3$, etc.

Let $[a, b]$ be a fixed closed interval, and let $J_1, \dots, J_n$ be n pairs of functions defined on subintervals. Suppose that J_k connects with J_{k+1} and that the endpoints of the corresponding intervals form a partition of $[a, b]$. We call

$$J_1 \odot J_2 \odot \cdots \odot J_n$$

a *chain over* $[a, b]$ having $J_1, \dots, J_n$ as its *links*. Let $\mathcal{J}$ be a family of pairs $J = ([c, d], g)$ as above, where $[c, d] \subset [a, b]$. The *chain envelope* $\varepsilon(\mathcal{J})$ of $\mathcal{J}$ consists of all possible chains over $[a, b]$ constructed from the elements of $\mathcal{J}$. Finally, let $\mathcal{N} \subset [a, b]$ and $\mathcal{M} \subset \mathbb{R}$. A family $\mathcal{J}$ of joins of continuous functions is called *complete with respect to* $\mathcal{M}$ *and* $\mathcal{N}$ if for every $a' < b' \in \mathcal{N}$ and every $c' \neq d' \in \mathcal{M}$, there exists a join $([a', b'], f) \in \mathcal{J}$ such that $f(a') = c'$ and $f(b') = d'$.

3.2. Density of $DNM[a, b]$ in $\mathcal{C}[a, b]$

The previous concepts will help us to prove some of our main results. In fact, a couple of simple facts will allow us to prove the denseness of $DNM[a, b]$ in $\mathcal{C}[a, b]$.

Lemma 3.1. *Let $\mathcal{N}$ be a dense subset of $[a, b]$ (containing both a and b) and $\mathcal{M}$ be a dense subset of $\mathbb{R}$, and consider a complete family $\mathcal{J}$ of k-rectangular pairs inside $[a, b]$ with respect to $\mathcal{N}$ and $\mathcal{M}$, where $k \in [1, +\infty)$ is fixed. Then $\varepsilon(\mathcal{J})$ is dense in $\mathcal{C}[a, b]$.*

Proof. Let $f \in \mathcal{C}[a, b]$ and $\varepsilon > 0$. There exists a finite sequence $a_0 = a < a_1 < a_2 < \cdots < a_n = b$ of elements of $\mathcal{N}$ such that $\mathrm{Osc}_{I_j} f < \frac{\varepsilon}{9k}$, where I_j denotes the interval $[a_j, a_{j+1}]$ for every $j = 0, 1, \dots, n-1$.

From the completeness of $\mathcal{J}$, there is a k-rectangular pair $J_0 = (I_0, f_0) \in \mathcal{J}$ with $|f_0(a_0) - f(a_0)| < \frac{\varepsilon}{9k}$ and $|f_0(a_1) - f(a_1)| < \frac{\varepsilon}{9k}$. With this choice we have that $|f_0(a_0) - f_0(a_1)| < \frac{\varepsilon}{3k}$. Therefore $\mathrm{Osc}_{I_0} f_0 < \frac{\varepsilon}{3}$, and thus $|f(t) - f_0(t)| < \varepsilon$ for every $t \in I_0$.

Now, there exists another k-rectangular pair $J_1 = (I_1, f_1) \in \mathcal{J}$ satisfying that $f_1(a_1) = f_0(a_1)$ and $|f_1(a_2) - f(a_2)| < \frac{\varepsilon}{9k}$. Like this we have that $|f_1(a_1) - f_1(a_2)| < \frac{\varepsilon}{3k}$, therefore $\mathrm{Osc}_{I_1} f_1 < \frac{\varepsilon}{3}$, and thus $|f(t) - f_1(t)| < \varepsilon$ for every $t \in I_1$.

By repeating this process, we finally deduce that there exists a k-rectangular pair $J_{n-1} = (I_{n-1}, f_{n-1}) \in \mathcal{J}$ such that $f_{n-1}(a_{n-1}) = f_{n-2}(a_{n-1})$ and

$$|f_{n-1}(a_n) - f(a_n)| < \frac{\varepsilon}{9k}.$$

In that situation we have that $|f_{n-1}(a_{n-1}) - f_{n-1}(a_n)| < \frac{\varepsilon}{3k}$, therefore

$$\mathrm{Osc}_{I_{n-1}} f_{n-1} < \frac{\varepsilon}{3},$$

and thus $|f(t) - f_{n-1}(t)| < \varepsilon$ for every $t \in I_{n-1}$.

At this point, define the chain

$$([a, b], \tilde{f}) := (I_0, f_0) \odot (I_1, f_1) \odot \cdots \odot (I_{n-1}, f_{n-1}) \in \varepsilon(\mathcal{J}).$$

We have that $|\tilde{f}(t) - f(t)| < \varepsilon$ for all $t \in [a, b]$, and we are done. □

Lemma 3.2. *For each set of four points $a, b, c, d \in \mathbb{R}$ with $a < b$, there exists a 1-rectangular pair $([a, b], h)$ with $h \in DNM[a, b]$, $h(a) = c$, $h(b) = d$, and $h'(a) = h'(b) = 0$.*

Proof. Take any function $f \in DNM[0, 1]$. Let $t_{\max}$ and $t_{\min}$ be points at which f attains, respectively, its absolute maximum and absolute minimum on $[0, 1]$. By restricting f to a strictly smaller subinterval, we may assume that $0 < t_{\max}, t_{\min} < 1$. Also, without loss we assume that $t_{\max} < t_{\min}$. Notice that $f'(t_{\max}) = f'(t_{\min}) = 0$, and the pair $\{[t_{\max}, t_{\min}], f\}$ is 1-rectangular.

Finally, since regularity and nowhere differentiability are properties invariant under affine transformations, we can set $h(t) = A + Bf(kt + l)$, where the four parameters A, B, k, l are chosen in order to make $\{[a, b], h\}$ satisfy the required properties. □

Theorem 3.3. *$DNM[a, b]$ is dense in $\mathcal{C}[a, b]$.*

Proof. Let $\mathcal{J}_{DNM}$ be the set of all pairs (I, f) that are k-rectangular for some k, where $I \subset [a, b]$ and where f is a differentiable nowhere monotone function on I whose derivative vanishes at the endpoints of I. According to Lemma 3.2, $\mathcal{J}_{DNM}$ is complete with respect to $\mathcal{N} = [a, b]$ and $\mathcal{M} = \mathbb{R}$. By Lemma 3.1, $\varepsilon(\mathcal{J}_{DNM})$ is dense in $\mathcal{C}[a, b]$. Finally, observe that, for every pair $(I, f) \in \mathcal{J}_{DNM}$, the derivative of f vanishes at the endpoints of I, and hence we can connect pairs of $\mathcal{J}_{DNM}$ to obtain a differentiable function. Therefore, $\varepsilon(\mathcal{J}_{DNM}) \subset DNM[a, b]$ and $DNM[a, b]$ is dense in $\mathcal{C}[a, b]$. □

4. Dense-lineability of *DNM*$[a, b]$

This section will be focused on the dense-lineability of the set of differentiable nowhere monotone functions on $[a, b]$, $DNM[a, b]$. Without loss, we will be working in the interval $[0, 1]$, but everything trivially extends to any closed interval $[a, b]$.

4.1. Some modifications of old results

In this subsection, we will state some straightforward modifications of results in [1]. First, an examination of the result, technique and proof of [1, Theorem 3.1] yields the following.

Proposition 4.1. *Let $0 < y_0 < y_1 < y_2 < \cdots < y_n < \cdots \to 1$. Let $\bar{\alpha} = \{\alpha_j\}_{j=1}^{\infty}$ and $\bar{\beta} = \{\beta_j\}_{j=1}^{\infty}$ be countable sets without common elements in $[0, 1]$. Then there exists a differentiable increasing function F on $[0, 1]$ with $F(0) = 0$, and such that $F'(\alpha_j) = 1$ for all $j \in \mathbb{N}$, $F'(\beta_j) = y_j$ for all $j \in \mathbb{N}$, and $0 < F'(x) \leq 1$, for all $x \in [0, 1]$. Moreover, given $\varepsilon > 0$, we can choose F so that either $F(1) > y_0 - \varepsilon$ or $F(1) < \varepsilon$.*

The following modification of [1, Theorem 3.2] now follows.

Proposition 4.2. *Let A^+, A^-, A^0 be pairwise disjoint countable sets in $[0, 1]$ with $\{0, 1\} \subset A^0$. There exists a differentiable 3-rectangular pair $([0, 1], H(x))$, $H(0) \neq H(1)$, having the following properties: $H'(x) > 0$, for $x \in A^+$, $H'(x) < 0$, for $x \in A^-$, and $H'(x) = 0$, for $x \in A^0$.*

Finally, from Proposition 4.2 and by making affine transformations, we deduce the following corollary.

Corollary 4.3. *Let $[a, b]$ be an interval and consider three pairwise disjoint countable dense subsets A^+, A^-, A^0 of $[a, b]$. For every $v \neq w \in \mathbb{R}$, there exists a differentiable nowhere monotone function f on $[a, b]$ satisfying that $f' > 0$ on A^+, $f' < 0$ on A^-, $f' = 0$ on A^0, $f(a) = v$ and $f(b) = w$, and the pair $([a, b], f)$ is 3-rectangular.*

4.2. Dyadic intervals and dense-lineability of *DNM*$[0, 1]$

We will deal with *dyadic intervals* $I_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$, $k \in \{0, 1, \ldots, 2^n - 1\}$, $n \in \mathbb{N}$. A pair (I, f) will be called dyadic if I is a dyadic interval and it will be called differentiable if f is differentiable. Let $\mathcal{J}$ be a family of dyadic pairs, and consider a subset $\mathcal{M}$ of $\mathbb{R}$. We will say that $\mathcal{J}$ is complete with respect to $\mathcal{M}$ if for every dyadic interval $I_{n,k}$ and every $v \neq v' \in \mathcal{M}$ there exists a dyadic pair $(I_{n,k}, f) \in \mathcal{J}$ such that $f(k/2^n) = v$ and $f((k+1)/2^n) = v'$.

We will split the proof of the dense-lineability of $DNM[0, 1]$ into several lemmas, relating joins, chains and dyadic intervals.

Lemma 4.4. *Let $\mathcal{M}$ be a dense set of $\mathbb{R}$ and $k \in [1, +\infty)$. Let $\mathcal{J}$ be a complete family of k-rectangular dyadic pairs with respect to $\mathcal{M}$. Let $\mathcal{H}$ be the subset of $\varepsilon(\mathcal{J})$ composed of all chains over $[0, 1]$ whose links are over dyadic intervals of the same order. Then $\mathcal{H}$ is dense in $\mathcal{C}[0, 1]$.*

Proof. Notice that, given any continuous function $f \in \mathcal{C}[0, 1]$ and any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ so large that the oscillation of f over each dyadic interval of order n is less than or equal to $\varepsilon/9k$. From this point, we can continue as in the proof of Lemma 3.1. $\square$

Lemma 4.5. *Let $\mathcal{J}$ be a countable family of differentiable dyadic pairs covering $[0, 1]$. Assume that for every dyadic pair $(I, f) \in \mathcal{J}$ there exist three dense sets $A^+_{I,f}, A^-_{I,f}, A^0_{I,f}$ in I such that $f'(x) > 0$ if $x \in A^+_{I,f}$, $f'(x) < 0$ if $x \in A^-_{I,f}$, $f'(x) = 0$ if $x \in A^0_{I,f}$, and the endpoints of I belong to $A^0_{I,f}$.*

Also, assume the following:

Given any dyadic interval I, the dyadic pairs of the form (I, f) can be relabeled, $\{(I, f_k) : k \in \mathbb{N}\}$, so that if $i > j$ then $A^+_{I,f_i} \cup A^-_{I,f_i} \cup A^0_{I,f_i} \subset A^0_{I,f_j}$.

Given any two dyadic pairs $(I, f), (J, g) \in \mathcal{J}$ with $I \subsetneq J$, then $A^+_{I,f} \cup A^-_{I,f} \cup A^0_{I,f} \subset A^0_{J,g}$.

Let $\mathcal{H}$ be the subset of $\varepsilon(\mathcal{J})$ composed of all chains over $[0, 1]$ whose links are dyadic intervals of the same order. Then, $\operatorname{span}(\mathcal{H}) \subset DNM[0, 1] \cup \{0\}$.

Proof. First of all, observe that for every dyadic pair $(I, f) \in \mathcal{J}$, the endpoints of I belong to $A^0_{I,f}$. Therefore the derivatives of f at the endpoints of I are 0, and thus we can connect two dyadic pairs in $\mathcal{J}$ and obtain a differentiable function.

Let $g = \sum_{k=1}^{m} \lambda_k g_k$ with $\lambda_1, \ldots, \lambda_m \in \mathbb{R} \setminus \{0\}$ and $g_1, \ldots, g_m \in \mathcal{H}$. We may assume without loss of generality that there are natural numbers $m_1, m_2, \ldots, m_p$ and $n_1, n_2, \ldots, n_p$ such that

1. $m_1 + \cdots + m_p = m$;
2. for every $0 \leq j \leq p - 1$ and every $m_j + 1 \leq k \leq m_{j+1}$ ($m_0 = 0$), the links of g_k are of order n_{j+1}; and
3. $n_1 > n_2 > \cdots > n_p$.

Our aim is to show that g is nowhere monotone on every dyadic interval of order n_1, which will complete the proof. Let then I be a dyadic interval of order n_1. Again, by hypothesis we may assume without loss that $A^+_{I,g_1} \cup A^-_{I,g_1} \cup A^0_{I,g_1} \subset A^0_{I,g_k}$ for every $2 \le k \le m_1$. Furthermore, we have that $A^+_{I,g_1} \cup A^-_{I,g_1} \cup A^0_{I,g_1} \subset A^0_{I,g_k}$ for every $m_1 + 1 \le k \le m$. Therefore, if $t \in A^+_{I,g_1} \cup A^-_{I,g_1} \cup A^0_{I,g_1}$, then $g'(t) = \lambda_1 g_1'(t)$, which shows that g is nowhere monotone on I. □

Lemma 4.6. *For every dyadic interval I there exist three sequences $(A^+_{I,n})_{n\in\mathbb{N}}$, $(A^-_{I,n})_{n\in\mathbb{N}}$, $(A^0_{I,n})_{n\in\mathbb{N}}$ with the following properties:*

1. *$A^+_{I,n}, A^-_{I,n}, A^0_{I,n}$ are pairwise disjoint countable dense subsets of I for every $n \in \mathbb{N}$.*
2. *The endpoints of I belong to $A^0_{I,n}$ for every $n \in \mathbb{N}$.*
3. *$A^+_{I,n} \cup A^-_{I,n} \cup A^0_{I,n} \subset A^0_{I,m}$ if $n > m$.*
4. *If J is another dyadic interval with $I \subsetneq J$, then $A^+_{I,n} \cup A^-_{I,n} \cup A^0_{I,n} \subset A^0_{J,m}$ for every $n, m \in \mathbb{N}$.*

Proof. Firstly, let $\phi : \mathbb{R} \longrightarrow (0, 1)$ be any homeomorphism. For every $n \in \mathbb{N}$, let $B_n = \phi(\mathbb{Q} + \pi n)$. We have that $(B_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint countable dense subsets of $(0, 1)$. Secondly, let P denote the set of prime numbers, and consider $\{B_{p^k} : p \in P, k \in \mathbb{N}\}$. We will describe the inductive process that needs to be followed in order to reach our goal. Let $C_2 = \bigcup\{B_{p^k} : p \in P, p \neq 2, k \in \mathbb{N}\}$. Next, consider the following sets:

(I.1) $A^+_{[0,1],1} = B_2$,
(I.2) $A^-_{[0,1],1} = B_{2^2} = B_4$,
(I.3) $A^0_{[0,1],1} = C_2 \cup \left(\bigcup_{k>2} B_{2^k}\right) \cup \{0, 1\}$.

Secondly, let

(II.1) $A^+_{[0,1],2} = B_{2^3} = B_8$,
(II.2) $A^-_{[0,1],2} = B_{2^4} = B_{16}$,
(II.3) $A^0_{[0,1],2} = C_2 \cup \left(\bigcup_{k>4} B_{2^k}\right) \cup \{0, 1\}$.

We can follow this process to obtain at the mth step

(m.1) $A^+_{[0,1],m} = B_{2^{2m-1}}$,
(m.2) $A^-_{[0,1],m} = B_{2^{2m}}$,
(m.3) $A^0_{[0,1],m} = C_2 \cup \left(\bigcup_{k>2m} B_{2^k}\right) \cup \{0, 1\}$.

Clearly, for every $n \in \mathbb{N}$, we have that $A^+_{[0,1],n} \cup A^-_{[0,1],n} \cup A^0_{[0,1],n} \subset A^0_{[0,1],m}$ if $n > m$. Now, we apply the same process to both $[0, 1/2]$ and $[1/2, 1]$. For instance, in the case of $[0, 1/2]$, call

$$C_3 = \bigcup\{B_{p^k} : p \in P, p > 3, k \in \mathbb{N}\}$$

and

(m.1) $A^+_{[0,1/2],m} = B_{3^{2m-1}} \cap [0, 1/2]$,
(m.2) $A^-_{[0,1/2],m} = B_{3^{2m}} \cap [0, 1/2]$,
(m.3) $A^0_{[0,1/2],m} = \left(C_3 \cup \left(\bigcup_{k>2m} B_{3^k}\right) \cup \{0, 1/2\}\right) \cap [0, 1/2]$.

Clearly, for every $n \in \mathbb{N}$, we have that $A^+_{[0,1/2],n} \cup A^-_{[0,1/2],n} \cup A^0_{[0,1/2],n} \subset A^0_{[0,1/2],m}$ if $n > m$. Moreover, $A^+_{[0,1/2],n} \cup A^-_{[0,1/2],n} \cup A^0_{[0,1/2],n} \subset A^0_{[0,1],m}$ for every n, m.

We can continue in the same way and induction takes care of finishing this process, proving the lemma. □

Now we are ready to state and prove the main result of this section.

Theorem 4.7. *$DNM[a, b]$ is dense-lineable.*

Proof. Without loss, we will work in the interval $[0, 1]$. By Lemma 4.6, for every dyadic interval I there exist three sequences $(A^+_{I,n})_{n\in\mathbb{N}}$, $(A^-_{I,n})_{n\in\mathbb{N}}$, $(A^0_{I,n})_{n\in\mathbb{N}}$ such that:

1. $A^+_{I,n}, A^-_{I,n}, A^0_{I,n}$ are pairwise disjoint countable dense subsets of I for every $n \in \mathbb{N}$.
2. The endpoints of I belong to $A^0_{I,n}$ for every $n \in \mathbb{N}$.
3. $A^+_{I,n} \cup A^-_{I,n} \cup A^0_{I,n} \subset A^0_{I,m}$ if $n > m$.
4. If J is another dyadic interval with $I \subsetneq J$, then $A^+_{I,n} \cup A^-_{I,n} \cup A^0_{I,n} \subset A^0_{J,m}$ for every $n, m \in \mathbb{N}$.

Now, let us consider the countable set $\{(v, w) \in \mathcal{M} \times \mathcal{M} : v \neq w\}$ where $\mathcal{M}$ is a countable dense subset of $\mathbb{R}$. Let us relabel this set as $(u_n)_{n\in\mathbb{N}}$. Next, by Corollary 4.3, for every dyadic interval I and every $n \in \mathbb{N}$, there exists a differentiable nowhere monotone function $f_{I,n}$ on I satisfying that:

1. $f'_{I,n} > 0$ on $A^+_{I,n}$.
2. $f'_{I,n} < 0$ on $A^-_{I,n}$.
3. $f'_{I,n} = 0$ on $A^0_{I,n}$.
4. $f_{I,n}$ maps, respectively, the endpoints of I to the coordinates of u_n.
5. The pair $(I, f_{I,n})$ is 3-rectangular.

Finally, $\mathcal{J} = \{(I, f_{I,n}) : I$ is a dyadic interval and $n \in \mathbb{N}\}$ is a complete family of 3-rectangular dyadic pairs with respect to $\mathcal{M}$ which satisfies the conditions of Lemma 4.5. By Lemmas 4.4 and 4.5, the result is proved. □

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